ЗК-19 11.11.20

Замена переменных в двойном интеграле

п1. Двойной интеграл в полярных координатах

x, y – прямоугольные координаты

r, φ – полярные координаты

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \quad (1)$$

$$\iint\limits_{D(x,y)} f(x,y)\,dx\,dy = \iint\limits_{D(r,\varphi)} f(r\cos\varphi, r\sin\varphi)\,\underline{|J|}\,dr\,d\varphi$$

J – якобиан $\quad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$

$x = r\cos\varphi$

$x'_r = \left.\frac{\partial x}{\partial r}\right|_{\varphi=const} = (r\cos\varphi)'_r = \cos\varphi$

$x'_\varphi = \left.\frac{\partial x}{\partial \varphi}\right|_{r=const} = (r\cos\varphi)'_\varphi = r(-\sin\varphi)$

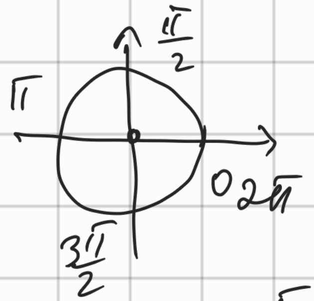
$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi - (-r\sin^2\varphi) = r\cos^2\varphi + r\sin^2\varphi = r(\cos^2\varphi + \sin^2\varphi) = r$$

$$\boxed{J = r}$$

$\underline{x^2 + y^2} = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\cos^2\varphi + \sin^2\varphi) = r^2$

Примеры

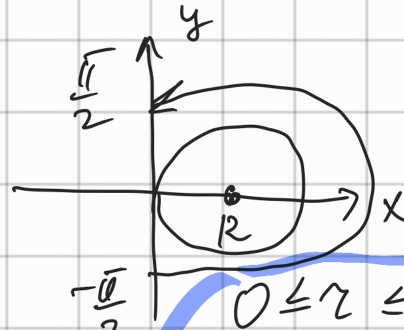
1) $x^2+y^2=R^2$



$0\le r\le R$

$0\le \varphi\le 2\pi$

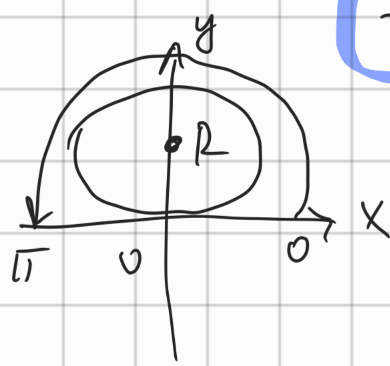
2) $r=R\cos\varphi$



$0\le r\le R\cos\varphi$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

3) $r=R\sin\varphi$



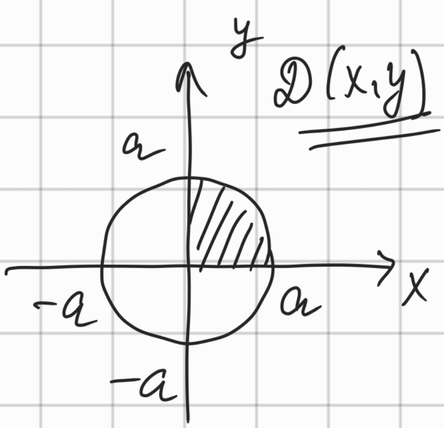
$0\le r\le R\sin\varphi$

$0\le\varphi\le\pi$

Задачи

1) $\iint\limits_{D}\sqrt{x^2+y^2}\,dxdy$, если

D – I четверть круга $x^2+y^2\le a^2$



$D(x,y)$

переходим к полярным координатам с помощью

$x=r\cos\varphi$

$y=r\sin\varphi$

$D:\ r^2\le a^2$

$0\le r\le a$

$0\le\varphi\le 2\pi$

$$\iint\limits_{D(x,y)}\sqrt{x^2+y^2}\,dxdy=\iint\limits_{D(r,\varphi)}\sqrt{r^2}\cdot r\,dr\,d\varphi=$$

$$= \int_0^{2\pi} d\varphi \int_0^{a} r^2\,dr = \varphi \Big|_0^{2\pi} \cdot \frac{r^3}{3}\Big|_0^{a} =$$

$$= (2\pi - 0)\left(\frac{a^3}{3} - 0\right) = \boxed{\frac{2\pi a^3}{3}}$$

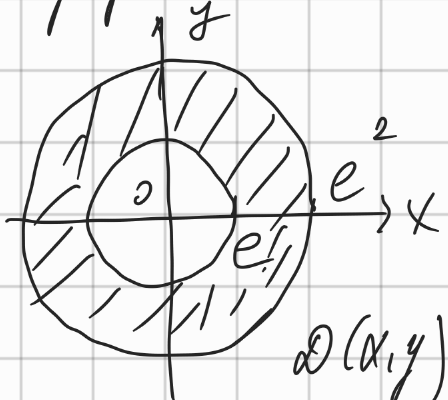
2) $\iint\limits_{D} \ln(x^2+y^2)\,dx\,dy$, D – кольцо между окружностями $x^2+y^2=e^2$ и $x^2+y^2=e^4$

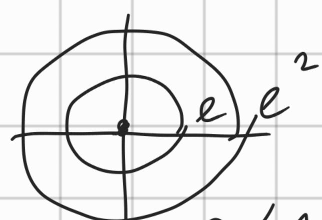
$x = r\cos\varphi \quad y = r\sin\varphi$

$D(r,\varphi)$

$r^2 = e^2 \qquad r = e$

$r^2 = e^4 \qquad r = e^2$





$e \le r \le e^2$

$0 \le \varphi \le 2\pi$

$$\iint\limits_{D(x,y)} \ln\overset{r^2}{(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\varphi \int_e^{e^2} \ln r^2 \cdot r\,dr = 2\int_0^{2\pi} d\varphi \cdot \int_e^{e^2} r\ln r\,dr =$$

$$= 2\cdot\varphi\Big|_0^{2\pi} \cdot \int_e^{e^2} r\ln r\,dr = 4\pi \cdot \underline{I}$$

$$\underline{I} = \int_e^{e^2} r\ln r\,dr = \left\{ \begin{aligned} &\ln r = u, && du = \frac{1}{r}dr \\ &r\,dr = dv, && v = \int r\,dr = \frac{r^2}{2} \end{aligned} \right\}$$

$$= \left[\int u\,dv = uv - \int v\,du \right] = \frac{r^2}{2}\ln r - \int \frac{r^2}{2}\cdot\frac{1}{r}\,dr =$$

$$= \frac{r^2}{2}\ln r - \frac{r^2}{4}\Bigg|_e^{e^2} = \frac{e^4}{2}\underbrace{\ln e^{2}}_{=1} - \frac{e^4}{4} -$$

$$-\left(\frac{e^2}{2}\underbrace{\ln e}_{=e} - \frac{e^2}{4}\right) = e^4 - \frac{e^4}{4} - \frac{e^2}{2} + \frac{e^2}{4} =$$

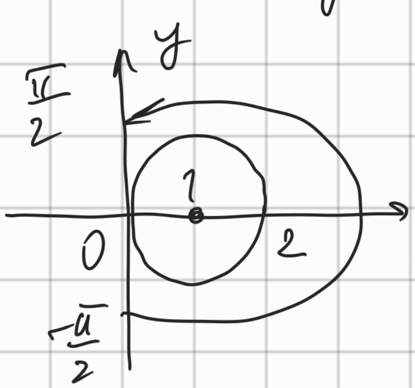
$$= \frac{3e^4}{4} - \frac{e^2}{4} = \frac{3e^4 - e^2}{4}$$

$$\iint\limits_D \ln(x^2+y^2)\,dx\,dy = 4\pi\cdot\frac{3e^4-e^2}{4} = \pi(3e^4 - e^2)$$

3) $\iint\limits_D (x^2+y^2)\,dx\,dy$ $\quad D: \; x^2+y^2 = 2x$

$$x^2 - 2x + y^2 = 0$$

$(x-1)^2 - 1 + y^2 = 0 \quad (x-1)^2 + y^2 = 1 \quad C(1;0)\; R=1$



$D(r,\varphi)$: $\quad x = r\cos\varphi \quad y = r\sin\varphi$

$$r^2 = 2r\cos\varphi$$

$$r = 2\cos\varphi$$

$$0 \le r \le 2\cos\varphi$$

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$\iint\limits_D (x^2+y^2)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\cdot r\,dr =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \left.\frac{r^4}{4}\right|_0^{2\cos\varphi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16\cos^4\varphi}{4} =$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\varphi)^2 d\varphi = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\varphi + \underbrace{\cos^2 2\varphi}\right) d\varphi = \int_{-\frac{\pi}{2}}^{\pi/2} \left(1+2\cos 2\varphi + \frac{1+\cos 4\varphi}{2}\right) d\varphi =$$

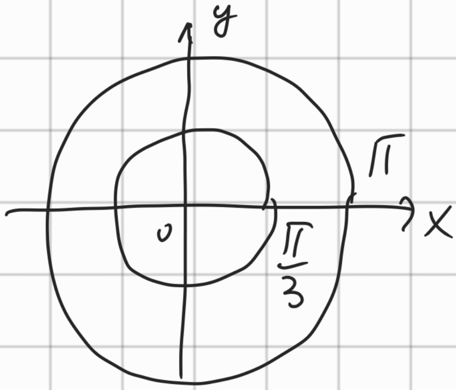
$$= \varphi + \cancel{2}\,\frac{\sin 2\varphi}{\cancel{2}} + \frac{1}{2}\left(\varphi + \frac{\sin 4\varphi}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \underbrace{\frac{\pi}{2}} + \overset{=0}{\sin\pi} + \underbrace{\frac{1}{2}\cdot\frac{\pi}{2}} + \frac{1}{8}\cdot\overset{=0}{\sin 2\pi} - \left(-\frac{\pi}{2} + \underset{0}{\underset{\parallel}{\sin(-\pi)}} + \frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{8}\underset{0}{\underset{\parallel}{\sin(-2\pi)}}\right) = \underbrace{\frac{\pi}{2}} + \underbrace{\frac{\pi}{4}} + \underbrace{\frac{\pi}{2}}$$

$$+ \underbrace{\frac{\pi}{4}} = \pi + \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$$

4) $\displaystyle\iint_D \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,dx\,dy \qquad D:\ x^2+y^2 \le \pi^2$

$\qquad\qquad x^2+y^2 \ge \frac{\pi^2}{9}$



$r^2=\pi^2$ $\quad r=\pi$

$r^2=\frac{\pi^2}{9}$ $\quad r=\frac{\pi}{3}$ $\quad \underline{\frac{\pi}{3}\le r\le\pi}$

$\underline{0\le\varphi\le 2\pi}$

$$\iint_D \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,dx\,dy =$$

$$=\int_0^{2\pi} d\varphi \int_{\pi/3}^{\pi} \frac{\sin r}{\cancel{r}}\cdot \cancel{r}\,dr = \varphi\Big|_0^{2\pi}\cdot(-\cos r)\Big|_{\pi/3}^{\pi} =$$

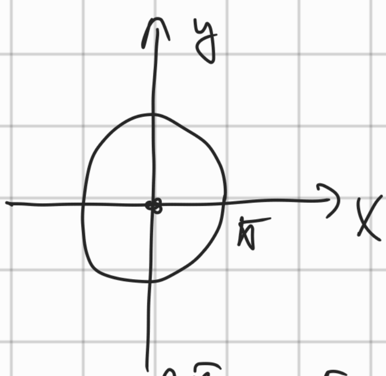
$$=2\pi\left(-\underbrace{\cos\pi}_{-1}+\cos\frac{\pi}{3}\right) = 2\pi\left(1+\frac{1}{2}\right)=2\pi\cdot\frac{3}{2}=\underline{3\pi}$$

5) $\iint_D \left(1-\frac{y^2}{x^2}\right)dx\,dy$ $\quad D:\ x^2+y^2\le\pi^2$

$r^2\le\pi^2$

$0\le r\le\pi$

$0\le\varphi\le 2\pi$



$$\iint_D \left(1-\frac{y^2}{x^2}\right)dx\,dy =$$

$$=\int_0^{2\pi} d\varphi\int_0^{\pi}\left(1-\frac{\cancel{r^2}\sin^2\varphi}{\cancel{r^2}\cos^2\varphi}\right)r\,dr =$$

$$=\int_0^{2\pi} d\varphi\int_0^{\pi}\left(1-tg^2\varphi\right)r\,dr = \int_0^{2\pi}\left(1-tg^2\varphi\right)d\varphi\int_0^{\pi} r\,dr$$

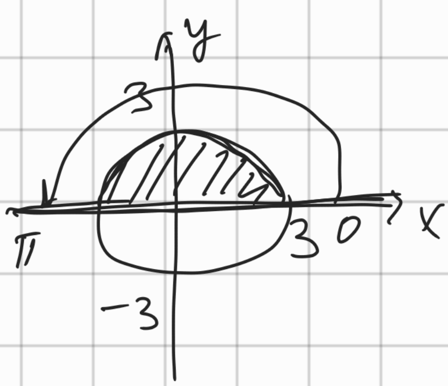
$$\left[1+tg^2\varphi=\frac{1}{\cos^2\varphi}\right] \quad tg^2\varphi=\frac{1}{\cos^2\varphi}-1$$

$$=\int_0^{2\pi}\left(1-\frac{1}{\cos^2\varphi}+1\right)d\varphi\int_0^{\pi} r\,dr=\int_0^{2\pi}\left(2-\frac{1}{\cos^2\varphi}\right)d\varphi\cdot$$

$$\cdot\frac{r^2}{2}\Big|_0^{\pi}=\left(2\varphi-tg\,\varphi\right)\Big|_0^{2\pi}\cdot\frac{\pi^2}{2}=(4\pi-\underbrace{tg\,2\pi}_{=0})\frac{\pi^2}{2}=$$

$$=4\pi\cdot\frac{\pi^2}{2}=\underline{2\pi^3}$$

6) $\iint\limits_D \frac{dx\,dy}{x^2+y^2+3}$ $\quad D=\{y=\sqrt{9-x^2},\ y=0\}$



$y=\sqrt{9-x^2}$

$y^2=9-x^2 \quad x^2+y^2=9$

$r^2=9 \quad r=3$

$0\le r\le 3$

$0\le\varphi\le\pi$

$$\iint\limits_D\frac{dx\,dy}{x^2+y^2+3}=\int_0^{\pi}d\varphi\,\frac{1}{2}\int_0^{3}\frac{2r}{r^2+3}\,dr=$$

$$=\varphi\Big|_0^{\pi}\cdot\frac{1}{2}\ln|r^2+3|\Big|_0^{3}=\pi\cdot\frac{1}{2}\left(\ln|12|-\ln 3\right)=$$

$$=\underline{\frac{\pi}{2}\ln 4}$$