ЗК-19 19.11.20

Замена переменных в двойном интеграле: двойной интеграл в криволинейных координатах

Пусть дан $\iint\limits_{D(x,y)} f(x,y)\,dx\,dy$, где функция $f(x,y)$ - непрерывна в замкнутой области $D(x,y)$. Перейдем к новым переменным по формулам

$$\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned}\right\},$$

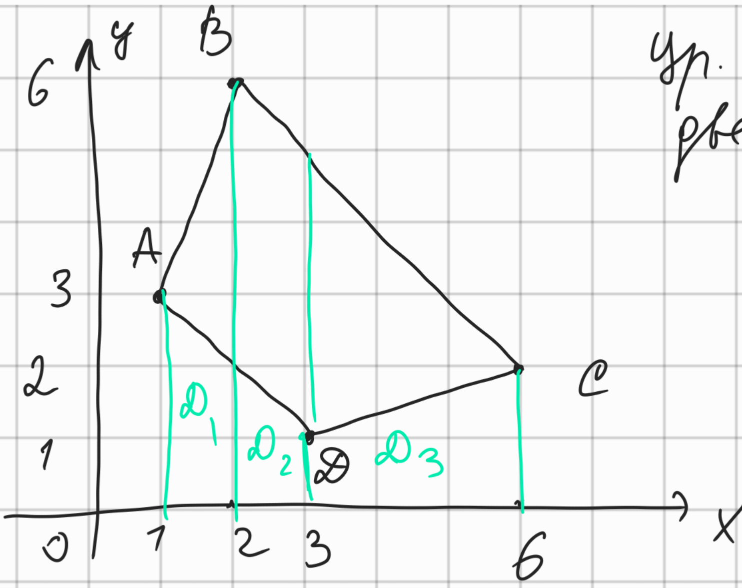
где функции новых переменных u и v непрерывны вместе со своими частными производными тогда справедлива формула замены переменных:

$$\iint\limits_{D(x,y)} f(x,y)\,dx\,dy = \iint\limits_{D(u,v)} f(x(u,v),y(u,v))\,\underline{\underline{|J|}}\,du\,dv$$

Определитель $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ наз-ся якобианом перехода от переменных x и y к переменным u и v

<u>Пример</u> 1) Вычислить $\iint\limits_{D} \frac{dx\,dy}{(x+y)^3}$

D: - Трапеция с вершинами $A(1;3)$ $B(2;6)$, $C(6;2)$, $D(3;1)$



Ур. пр., проходящей через две точки

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

Найдем ур. сторон трапеции

AB: $\frac{x-1}{2-1}=\frac{y-3}{6-3}$ $\quad \frac{x-1}{1}=\frac{y-3}{3}$ $\quad 3x-3=y-3$

$\boxed{3x-y=0}$ AB.

BC: $\frac{x-2}{6-2}=\frac{y-6}{2-6}$ $\quad \frac{x-2}{4}=\frac{y-6}{-4}$ $\quad -x+2=y-6$

$\boxed{x+y-8=0}$ BC.

CD: $\frac{x-6}{3-6}=\frac{y-2}{1-2}$ $\quad \frac{x-6}{-3}=\frac{y-2}{-1}$ $\quad x-6=3y-6$

$\boxed{x-3y=0}$ CD

AD: $\frac{x-1}{3-1}=\frac{y-3}{1-3}$ $\quad \frac{x-1}{2}=\frac{y-3}{-2}$ $\quad -x+1=y-3$

$\boxed{x+y-4=0}$ AD

Вводим новые переменные по формулам:

$$\boxed{\begin{aligned} x+y&=u \\ \frac{y}{x}&=v \end{aligned}}$$

$y=3x \rightarrow \frac{y}{x}=3$

$3y=x \rightarrow \frac{y}{x}=\frac{1}{3}$

$$\left.\begin{aligned} 4\le u\le 8 \\ \frac{1}{3}\le v\le 3 \end{aligned}\right\} D(u,v)$$

найти $x=$

$y=$

$$\begin{cases} x+y=u \\ y=vx \end{cases} \qquad \begin{aligned} x+vx&=u \\ x(1+v)&=u \end{aligned} \qquad \begin{aligned} x&=\frac{u}{1+v} \\ y&=\frac{uv}{1+v} \end{aligned}$$

$$\left.\begin{aligned} x &= \frac{u}{1+v} \\ y &= \frac{uv}{1+v} \end{aligned}\right\} \qquad \left.\frac{\partial x}{\partial u}\right|_{v=const} = \frac{1}{1+v}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}\cdot x' \qquad \frac{\partial x}{\partial v} = u\left(-\frac{1}{(1+v)^2}\right) = -\frac{u}{(1+v)^2}$$

$$\frac{\partial y}{\partial u} = \frac{v}{1+v} \qquad \left.\frac{\partial y}{\partial v}\right|_{u=c} = \frac{u(1+v)-uv}{(1+v)^2} = \frac{u+\cancel{uv}-\cancel{uv}}{(1+v)^2} =$$

$$= \frac{u}{(1+v)^2}$$

$$J = \begin{vmatrix} \frac{1}{1+v} & -\frac{u}{(1+v)^2} \\ \frac{v}{(1+v)} & \frac{u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^3} + \frac{uv}{(1+v)^3} =$$

$$= \frac{u\cancel{(1+v)}}{(1+v)^{\cancel{3}}} = \frac{u}{(1+v)^2}$$

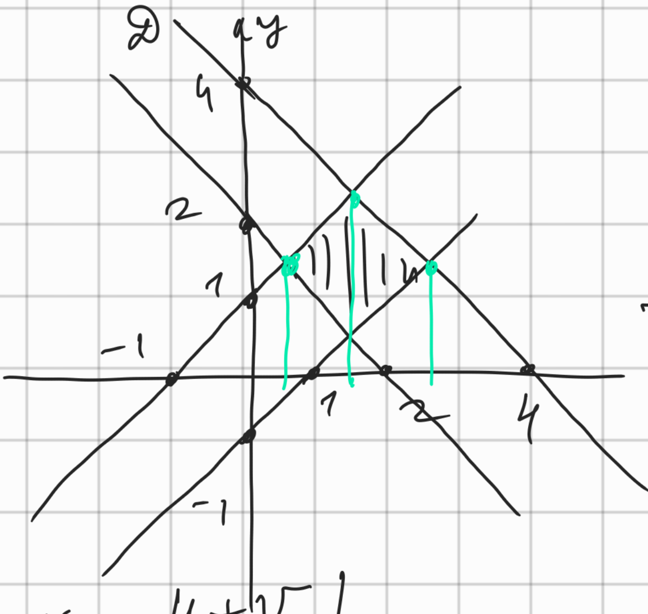
$$\iint\limits_D \frac{dx\,dy}{\underbrace{(x+y)^3}_{u}} = \int_4^8 du \int_{\frac{1}{3}}^{3} \frac{1}{u^{\cancel{3}\,2}}\cdot\frac{\cancel{u}}{(1+v)^2}\,dv =$$

$$\int_4^8 \frac{1}{u^2}\,du \int_{\frac{1}{3}}^{3} \frac{1}{(1+v)^2}\,dv = \left.\left(-\frac{1}{u}\right)\right|_4^8 \left.\left(-\frac{1}{1+v}\right)\right|_{\frac{1}{3}}^{3} =$$

$$= \left(\frac{1}{8}-\frac{1}{4}\right)\cdot\left(\frac{1}{4}-\frac{3}{4}\right) = \frac{1-2}{8}\cdot\left(\frac{-2}{4}\right) = \frac{-1}{8}\cdot\left(\frac{-1}{2}\right) = \frac{1}{16}$$

2) $\iint\limits_{D} (x+y)^3 (x-y)^2 \, dx\, dy$

$$D = \left\{ \begin{array}{l} x+y=2 \\ x-y=1 \\ x+y=4 \\ x-y=-1 \end{array} \right.$$



$$+\begin{cases} x+y=u \\ x-y=v \end{cases} \qquad \begin{array}{l} 2 \le u \le 4 \\ -1 \le v \le 1 \end{array}$$

$$2x = u+v$$

$$x = \frac{u+v}{2}$$

$$-\begin{array}{l} x+y=u \\ x-y=v \end{array}$$

$$2y = u-v \qquad y = \frac{u-v}{2}$$

$$\left. \begin{array}{l} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{array} \right\}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial x}{\partial v} = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

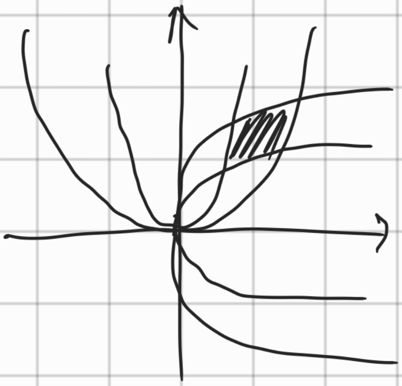
$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\iint\limits_{D} \underbrace{(x+y)}_{u}{}^3 \underbrace{(x-y)}_{v}{}^2 \, dx\, dy = \int_2^4 du \int_{-1}^{1} u^3 v^2 \frac{1}{2} \, dv =$$

$$= \frac{1}{2} \int_2^4 u^3 \, du \int_{-1}^{1} v^2 \, dv = \frac{1}{2} \left. \frac{u^4}{4} \right|_2^4 \left. \frac{v^3}{3} \right|_{-1}^{1} =$$

$$= \frac{1}{2} (64-4) \cdot \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{1}{2} \cdot 60 \cdot \frac{2}{3} = \underline{20},$$

3) $\iint\limits_{D} \frac{x^2 \sin\left(\frac{xy}{2}\right)}{y}\, dx\, dy$ $D: \begin{cases} x^2 = \frac{\pi y}{2}, & y^2 = 2x \\ x^2 = \frac{2\pi y}{3}, & y^2 = 4x \end{cases}$



$x^2 = \frac{\pi y}{2}$ $\frac{x^2}{y} = \frac{\pi}{2}$

$x^2 = \frac{2\pi y}{3}$ $\frac{x^2}{y} = \frac{2\pi}{3}$

$\boxed{\frac{x^2}{y} = u}$ $\frac{\pi}{2} \le u \le \frac{2\pi}{3}$

$y^2 = 2x$ $\frac{y^2}{x} = 2$ $\boxed{\frac{y^2}{x} = v}$ $2 \le v \le 4$

$y^2 = 4x$ $\frac{y^2}{x} = 4$

$\begin{cases} \frac{x^2}{y} = u \\ \frac{y^2}{x} = v \end{cases}$ $uy = x^2$ $y = \frac{x^2}{u}$

$\frac{x^4}{u^2 \cancel{x}} = v$ $x^3 = u^2 v$ $x = \sqrt[3]{u^2 v} = (u^2 v)^{\frac{1}{3}} = u^{\frac{2}{3}} v^{\frac{1}{3}}$

$\begin{cases} x = u^{\frac{2}{3}} v^{\frac{1}{3}} \\ y = u^{\frac{1}{3}} v^{\frac{2}{3}} \end{cases}$ $y = \frac{\left(u^{\frac{2}{3}} v^{\frac{1}{3}}\right)^2}{u} = \frac{u^{\frac{4}{3}} v^{\frac{2}{3}}}{u} = u^{\frac{1}{3}} v^{\frac{2}{3}}$

$\boxed{(x^n)' = n x^{n-1}}$

$\frac{\partial x}{\partial u} = v^{\frac{1}{3}} \frac{2}{3} u^{-\frac{1}{3}}$

$\frac{\partial x}{\partial v} = u^{\frac{2}{3}} \frac{1}{3} v^{-\frac{2}{3}}$

$\frac{\partial y}{\partial u} = v^{\frac{2}{3}} \cdot \frac{1}{3} u^{-\frac{2}{3}}$

$\frac{\partial y}{\partial v} = u^{\frac{1}{3}} \cdot \frac{2}{3} v^{-\frac{1}{3}}$

$$J=\begin{vmatrix} v^{\frac{1}{3}}\frac{2}{3}u^{-\frac{1}{3}} & u^{\frac{2}{3}}\frac{1}{3}v^{-\frac{2}{3}} \\ \frac{1}{3}v^{\frac{2}{3}}u^{-\frac{2}{3}} & \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}} \end{vmatrix}=$$

$$=\frac{4}{9}v^{\frac{1}{3}}u^{-\frac{1}{3}}u^{+\frac{1}{3}}v^{-\frac{1}{3}}-\frac{1}{9}v^{\frac{2}{3}}u^{-\frac{2}{3}}u^{\frac{2}{3}}v^{-\frac{2}{3}}=$$

$$=\frac{4}{9}-\frac{1}{9}=\frac{3}{9}=\frac{1}{3}$$

$$\iint_D \frac{x^2\sin\left(\frac{xy}{2}\right)}{y}\,dx\,dy=\frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}du\int_2^4 u\cdot\sin\left(\frac{u^{\frac{2}{3}}v^{-\frac{1}{3}}\cdot u^{\frac{1}{3}}v^{\frac{2}{3}}}{2}\right)dv$$

$$=\frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}u\,du\int_2^4\sin\left(\frac{uv}{2}\right)dv=\frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\cancel{u}\,du\left(-\cos\frac{uv}{2}\right.\cdot$$

$$\cdot\frac{2}{\cancel{u}}\Bigg|_2^4 dv=-\frac{2}{3}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\cos\frac{4u}{2}-\cos\frac{2u}{2}\right)du=$$

$$=-\frac{2}{3}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}(\cos 2u-\cos u)\,du=-\frac{2}{3}\left(\frac{\sin 2u}{2}-\sin u\right)\Bigg|_{\frac{\pi}{2}}^{\frac{2\pi}{3}}=$$

$$=-\frac{2}{3}\left(\frac{1}{2}\sin 2\cdot\frac{2\pi}{3}-\sin\frac{2\pi}{3}-\frac{1}{2}\underbrace{\sin \cancel{2}\cdot\frac{\pi}{\cancel{2}}}_{0}+\underbrace{\sin\frac{\pi}{2}}_{1}\right)=$$

$$=-\frac{2}{3}\left(\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}+1\right)=$$

$\sin\frac{4\pi}{3} = \sin 240^\circ \overset{-}{=} \sin 60^\circ = -\frac{\sqrt{3}}{2}$

$\sin\frac{2\pi}{3} = \sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

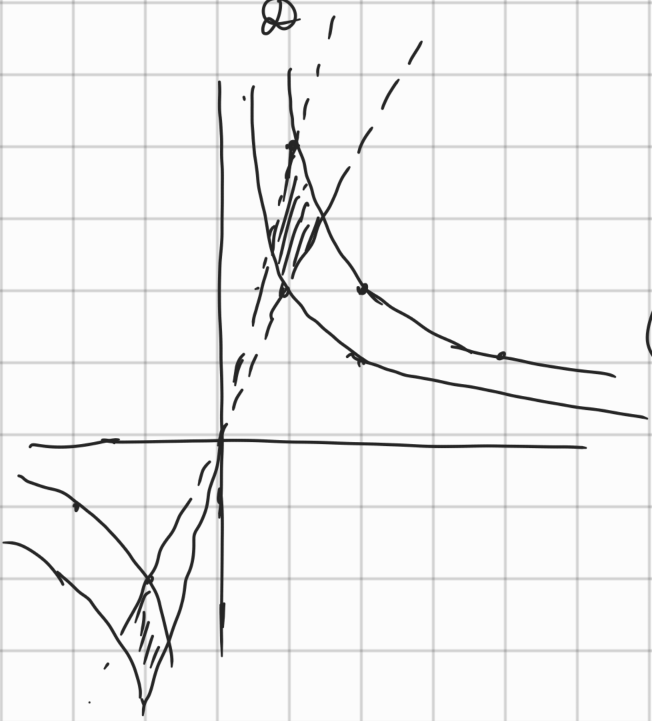
$= -\frac{2}{3}\left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + 1\right) =$

$= -\frac{2}{3}\left(-\frac{3\sqrt{3}}{4} + 1\right) =$

$= \underline{\frac{\sqrt{3}}{2} - \frac{2}{3}}$

4) $\iint\limits_{D} \frac{x}{y}\,dx\,dy$ $\qquad D: \begin{cases} xy=2 & xy=4 \\ y=2x & y=4x \end{cases}$

$\frac{y}{x}=2 \quad \frac{y}{x}=4 \quad y=\frac{2}{x} \quad y=\frac{4}{x}$



$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \qquad 2 \le u \le 4 \qquad 2 \le v \le 4$

$\frac{x}{y} = \frac{1}{v}$

$\begin{cases} y = xv \\ u = x^2 v \end{cases} \rightarrow x^2 = \frac{u}{v} \quad x = \sqrt{\frac{u}{v}} =$

$= \frac{u^{\frac{1}{2}}}{v^{\frac{1}{2}}} = u^{\frac{1}{2}} \cdot v^{-\frac{1}{2}}$

$\begin{cases} x = u^{\frac{1}{2}} v^{-\frac{1}{2}} \\ y = u^{\frac{1}{2}} v^{\frac{1}{2}} \end{cases}$

$y = u^{\frac{1}{2}} \underbrace{v^{-\frac{1}{2}} \cdot v}_{} = u^{\frac{1}{2}} v^{\frac{1}{2}}$

$\frac{\partial x}{\partial u} = \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} \qquad \frac{\partial x}{\partial v} = u^{\frac{1}{2}}\left(-\frac{1}{2}\right) v^{-\frac{3}{2}}$

$\frac{\partial y}{\partial u} = \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} \qquad \frac{\partial y}{\partial v} = u^{\frac{1}{2}} \frac{1}{2} v^{-\frac{1}{2}}$

$$J = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} =$$

$$= \frac{1}{4}u^{-\frac{1}{2}}v^{-\frac{1}{2}}u^{\frac{1}{2}}v^{-\frac{1}{2}} + \frac{1}{4}u^{\frac{1}{2}}v^{-\frac{3}{2}}u^{-\frac{1}{2}}v^{\frac{1}{2}} =$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1} = \frac{1}{2}v^{-1} = \frac{1}{2v}$$

$$\iint\limits_{D} \frac{x}{y}\,dx\,dy = \int_2^4 du \int_2^4 \frac{1}{2v^2}\,dv = \frac{1}{2}u\Big|_2^4 \left(-\frac{1}{v}\right)\Big|_2^4 =$$

$$= \frac{1}{2}(4-2)\left(-\frac{1}{4}+\frac{1}{2}\right) = \cancel{\frac{1}{2}}\cdot\cancel{2}\left(\frac{-1+2}{4}\right) = \frac{1}{4}$$